MATH2010E TUT010

Taylor's Formula for $f(x, y)$ at the Point (a, b) So $f \in C^{n+1}$

Suppose $f(x, y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$\begin{aligned} f(a+h, b+k) = {} & \underbrace{f(a, b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)} + \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\bigg|_{(a,b)}}_{P_n(h,k)} \\ & + \underbrace{\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\bigg|_{(a+ch,\, b+ck)}}_{\text{remainder}}. \end{aligned} \tag{7}$$

Lagrange form

The first n derivative terms are evaluated at (a, b). The last term is evaluated at some point $(a+ch, b+ck)$ on the line segment joining (a, b) and $(a+h, b+k)$.

So $c \in (0,1)$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = hf_x + kf_y$$

$$\begin{aligned} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f\right] = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)(hf_x + kf_y) \\ &= h\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f_x + k\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f_y \\ &= h^2 f_{xx} + hkf_{xy} + hkf_{yx} + k^2 f_{yy} \end{aligned}$$

If $(a, b) = (0, 0)$ and we treat h and k as independent variables (denoting them now by x and y), then Equation (7) assumes the following form.

Taylor's Formula for $f(x, y)$ at the Origin

$$\begin{aligned} f(x, y) = {} & f(0,0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) \\ & + \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y\frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right) \\ & + \frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\bigg|_{(cx,\, cy)} \end{aligned} \tag{8}$$

The first n derivative terms are evaluated at $(0, 0)$. The last term is evaluated at a point on the line segment joining the origin and (x, y).

Taylor's formula provides polynomial approximations of two-variable functions. The first n derivative terms give the polynomial; the last term gives the approximation error. The first three terms of Taylor's formula give the function's linearization. To improve on the linearization, we add higher-power terms.

Finding Quadratic and Cubic Approximations

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of f near the origin.

1. $f(x, y) = xe^y$ **2.** $f(x, y) = e^x \cos y$

Ans: $f_x = e^y$, $f_y = xe^y$

$f_{xx} = 0$, $f_{xy} = f_{yx} = e^y$, $f_{yy} = xe^y$

Quadratic approximation:

$$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right)$$

$$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right)$$

$$= x + xy$$

$f_{xxx} = 0$, $f_{xxy} = 0$, $f_{xyy} = e^y$, $f_{yyy} = xe^y$

Cubic approximation:

$$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right) + \frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right)$$

$$= x + xy + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 1 + y^3 \cdot 0\right)$$

$$= x + xy + \frac{1}{2}xy^2$$

9. $f(x, y) = \dfrac{1}{1 - x - y}$ **10.** $f(x, y) = \dfrac{1}{1 - x - y + xy}$

Ans: Recall $\dfrac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$ for $|t| < 1$

For (x,y) near $(0,0)$, we have $|x+y| \le |x| + |y| < 1$ and hence

$$f(x,y) = \frac{1}{1-(x+y)}$$

$$= 1 + (x+y) + (x+y)^2 + (x+y)^3 + \cdots$$

Quadratic approx:

$$f(x,y) \approx p_2(x,y) := 1 + (x+y) + (x+y)^2$$

Cubic approx:

$$f(x,y) \approx p_3(x,y) := 1 + (x+y) + (x+y)^2 + (x+y)^3$$

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12. Use Taylor's formula to find a quadratic approximation of $e^x \sin y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

Ans: Let $f(x,y) = e^x \sin y$

For (x,y) near $(0,0)$,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin y = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\Rightarrow f(x,y) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(y - \frac{y^3}{3!} + \cdots\right)$$

$$= y + xy + \text{higher order terms.}$$

Hence $f(x,y) \approx p_2(x,y) := y + xy$

Now, $f_x = e^x \sin y$, $f_y = e^x \cos y$

$f_{xx} = e^x \sin y$, $f_{xy} = e^x \cos y$, $f_{yy} = -e^x \sin y$

$f_{xxx} = e^x \sin y$, $f_{xxy} = e^x \cos y$, $f_{xyy} = -e^x \sin y$, $f_{yyy} = -e^x \cos y$ } all cts

By Taylor Thm, (Lagrange form)

$$f(x,y) - p_2(x,y) = \frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(cx,cy)}$$

for some $c \in (0,1)$

$\Rightarrow$ If $|x| \le 0.1$, $|y| \le 0.1$,

$$\text{Error} = |f(x,y) - p_2(x,y)|$$

$$\le \frac{1}{6}\Big[(0.1)^3 |e^{cx} \sin cy| + 3(0.1)^3 |e^{cx} \cos cy| + 3(0.1)^3 |e^{cx} \sin cy| + (0.1)^3 |e^{cx} \cos cy|\Big]$$

$$\le \frac{1}{6}(0.1)^3 e^{0.1}(1 + 3 + 3 + 1)$$

$$= \frac{4}{3}(0.1)^3 e^{0.1}$$

$$\approx 0.00147356$$

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44. The discriminant $f_{xx}f_{yy} - f_{xy}{}^2$ is zero at the origin for each of the following functions, so the Second Derivative Test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x, y)$ looks like. Describe your reasoning in each case.

a. $f(x, y) = x^2y^2$ **b.** $f(x, y) = 1 - x^2y^2$

c. $f(x, y) = xy^2$ **d.** $f(x, y) = x^3y^2$

e. $f(x, y) = x^3y^3$ **f.** $f(x, y) = x^4y^4$

Ans:

a) $f_x = 2xy^2$ $f_y = 2x^2y$

$\Rightarrow (0,0)$ is a critical pt

$f_{xx} = 2y^2$ $f_{xy} = 4xy$ $f_{yy} = 2x^2$

$\Rightarrow$ At $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0$ 2nd Derivative Test fails!

Note $f(x,y) = x^2y^2 > 0 = f(0,0)$ $\forall (x,y) \neq (0,0)$

$\Rightarrow f$ has a min. at $(0,0)$

c) $f_x = y^2$, $f_y = 2xy$

$\Rightarrow (0,0)$ is a critical pt

$f_{xx} = 0$ $f_{xy} = 2xy$, $f_{yy} = 2x$

$\Rightarrow$ At $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0$ 2nd Derivative Test fails!

Note $f(x,y) = xy^2 < 0 = f(0,0)$ if $x < 0$ and $y \neq 0$

> 0 if $x > 0$

$\Rightarrow f$ has neither a max or min at $(0,0)$ ✓